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P.R.-REGULATED SYSTEMS OF NOTATION AND THE SUBRECURSIVE HIERARCHY EQUIVALENCE PROPERTY(1)

BY

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ABSTRACT. We can attempt to extend the Grzegorczyk Hierarchy transfinitely by defining a sequence of functions indexed by the elements of a system of notation $\mathbb{S}$, using either iteration (majorization) or enumeration techniques to define the functions. (The hierarchy is then the sequence of classes of functions elementary in the functions of the sequence of functions.) In this paper we consider two sequences $\{F_s\}_{s\in\mathbb{S}}$ and $\{G_s\}_{s\in\mathbb{S}}$ defined by iteration and a sequence $\{E_s\}_{s\in\mathbb{S}}$ defined by enumeration; the corresponding hierarchies are $\{\mathcal{F}_s\}$, $\{\mathcal{G}_s\}$, $\{\mathcal{E}_s\}$. We say that $\mathbb{S}$ has the subrecursive hierarchy equivalence property if these two conditions hold:

(I) $\mathcal{E}_s = \mathcal{F}_s = \mathcal{G}_s$ for all $s \in \mathbb{S}$;

(II) $\mathcal{E}_s = \mathcal{E}_t$ for all $s, t \in \mathbb{S}$ such that $|s| = |t|$ ($|s|$ is the ordinal denoted by s).

We show that a certain type of system of notation, called p.r.-regulated, has the subrecursive hierarchy equivalence property. We present a nontrivial example of such a system of notation, based on Schütte's Klammersymbols. The resulting hierarchy extends those previously in print, which used the so-called standard fundamental sequences for limits $< \varepsilon_0$.

1. **Introduction.** The purpose of subrecursive hierarchies is to classify interesting classes of general recursive functions (perhaps ultimately all general recursive functions), according to "complexity", with a strictly expanding nest of sets indexed by ordinals. This is done by defining a sequence of progressively more complex functions, and then forming from each function the least class closed under certain operations. The two basic techniques which have been used to define the underlying sequence of functions are iteration (first found in Grzegorczyk [4]) and enumeration (Kleene [6]). Both techniques rely on diagonalization at limit ordinals λ. For example, if a sequence of functions $G_n(x)$ has been defined for all $n < \omega$, one might define $G_\omega(x) = G_x(x)$. In general, the diagonalization at limit λ requires the choice of a fundamental sequence $\{\lambda[n]\}_{n<\omega}$ for λ, i.e., a strictly increasing sequence of ordinals whose limit is λ. In the example given, $G_\omega(x) = G_{\omega[x]}(x)$, where $\omega[x] = x$.

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The choice of fundamental sequences is arbitrary. To avoid this, one might consider, say, all "primitive recursive" fundamental sequences, using systems of notation (Kleene [5]) to make this precise. If the classes so formed were independent of the choice of fundamental sequences, the technique would be justified. Axt [1], however, found that this fails at ω^2, and Feferman [2] found an even more radical "breakdown" of the hierarchy at ω^2.

In view of the breakdown of the hierarchy, researchers have fallen back on making unique choices of fundamental sequences, confining themselves to the so-called "standard" fundamental sequences. The resulting hierarchies provide convincing classifications of the n-recursive functions (Robbin [8]) and the functions primitive recursive in all finite types (Schwichtenberg [12]–also known as the ordinal recursive functions, Wainer [14]). Thus far the hierarchies in print extend over ε_0, but Wainer [15] states that he has extended them up to Γ_0, using the fundamental sequences in Feferman [3].

The notation of standard fundamental sequences for larger ordinals is progressively more obscure. While we can see fairly natural choices for some distance, there seems to be little point in further ad hoc methods. Rather, the time has come to seek a general theory of subrecursive hierarchies. The first step in such a program would be to seek conditions on systems of notation sufficient to prevent the breakdown of the associated hierarchy, yet weak enough to admit the accepted standard fundamental sequences. This paper presents such a condition, called "p.r.-regulated", which guarantees the "subrecursive hierarchy equivalence property" (essentially, that the iteration and enumeration hierarchies defined on the system of notation coincide and are independent of fundamental sequences at each ordinal level).

In §2 we treat preliminary matters of encoding and standard fundamental sequences. In §3 we define p.r.-regulated systems of notation, and present a nontrivial example utilizing Schütte's Klammersymbols (this system embraces the standard fundamental sequences). In §4 we define the subrecursive hierarchy equivalence property and prove that every p.r.-regulated system of notation satisfies this property. Finally, in §5 we outline areas of further research which this result makes possible.

2. **Preliminaries.** $\mathfrak{N}$ is the set of natural numbers $\{0, 1, 2, 3, \ldots\}$. By function, when the domain and range are not specified, we mean a function f: $\mathfrak{N}^n \to \mathfrak{N}$, where n is a positive integer. A partial function is a function f: $\mathfrak{K} \to \mathfrak{N}$, where $\mathfrak{K} \subseteq \mathfrak{N}^n$ for some positive integer n.

The class of primitive recursive functions is denoted $\mathcal{P}$. The class of elementary functions (Grzegorczyk [4]) is denoted $\mathcal{E}$. If $\mathcal{S}$ is a class of functions, then $\mathcal{E}(\mathcal{S})$ is the class of functions elementary in $\mathcal{S}$. If $\mathcal{S} = \{f_1, \ldots, f_m\}$ is a finite set of functions, we also write $\mathcal{E}(f_1, \ldots, f_m)$. $\{\mathcal{E}^n\}_{n \in \mathfrak{N}}$ is the Grzegorczyk Hierarchy [4].

If $Q(x_1, \ldots, x_n)$ is a relation (or predicate), then the characteristic function of Q is defined by

$$\mathrm{Char}_Q(x_1, \ldots, x_n) = \begin{cases} 1 & \text{iff } Q(x_1, \ldots, x_n), \\ 0 & \text{iff not } Q(x_1, \ldots, x_n), \end{cases}$$

Q is said to be *elementary in* $\mathbb{S}$ if $\mathrm{Char}_Q \in \mathcal{E}(\mathbb{S})$.

The function f defined by

$$f(x_1, \ldots, x_n) = \begin{cases} \text{the least } z < x_1 \text{ such that } Q(z, x_2, \ldots, x_n) \\ \qquad\qquad\qquad\qquad\qquad \text{if such } z \text{ exists,} \\ 0 \quad \text{otherwise,} \end{cases}$$

is said to be defined by *limited minimization*, and is denoted by

$$f(x_1, \ldots, x_n) = (\mu z < x_1) Q(x_1, \ldots, x_n).$$

If f is a unary function, the *iterates* of f are defined by the primitive recursion

$$f^0(x) = x, \quad f^{n+1}(x) = f(f^n(x)).$$

In §4, following Schwichtenberg [10], we will use the following encoding $\langle x_0, \ldots, x_n \rangle$ of finite sequences of natural numbers $x_0, \ldots, x_n$. A sequence $z_m, \ldots, z_0$ is a *modified binary representation* of x if each z_i $(0 \leqslant i \leqslant m)$ is either 1 or 2 and $x = 2^m z_m + 2^{m-1} z_{m-1} + \cdots + 2z_1 + z_0$. The empty sequence is taken as the modified binary representation of 0. Thus each natural number x has a unique modified binary representation $b(x)$, which we shall write as a concatenation of digits $b(x) = z_m z_{m-1} \cdots z_0$. For example, $b(3) = 11$, $b(5) = 21$, and $b(0)$ is the empty string.

The *ternary value* of a string of digits $z_m z_{m-1} \cdots z_0$ is $t(z_m z_{m-1} \cdots z_0) = 3^m z_m + 3^{m-1} z_{m-1} + \cdots + 3z_1 + z_0$. The ternary value of the empty string is 0. Now let

$$\langle x_0, x_1, \ldots, x_n \rangle = t(b(x_n)0b(x_{n-1})0 \cdots 0b(x_0)).$$

For example, $\langle 3, 0, 5 \rangle = t(b(5)0b(0)0b(3)) = t(210011) = 571$. The properties of this encoding which we will need are summarized in the following lemma, most of which is taken from Schwichtenberg [10].

LEMMA 1. (i) $\sum_{i=0}^{n} 3^i x_i \leqslant \langle x_0, x_1, \ldots, x_n \rangle \leqslant 3^n \prod_{i=0}^{n} (x_i + 1)^2$.

(ii) *If* $x \neq 0$ *then there are unique* k *and unique* $x_0, x_1, \ldots, x_k$ *such that* $x = \langle x_0, x_1, \ldots, x_k \rangle$ *and* $x_k \neq 0$.

(iii) *The following functions are elementary*:

$$\mathrm{In}_{n+1}(x_0, x_1, \dots, x_n) = \langle x_0, \dots, x_n \rangle,$$
$$\mathrm{Out}(x, i) = (x)_i = x_i \quad \textit{where } x = \langle x_0, x_1, \dots, x_i, \dots \rangle,$$
$$\begin{aligned} \mathrm{Ln}(x) &= n + 1 \quad \textit{if } x = \langle x_0, x_1, \dots, x_n \rangle \textit{ and } x_n \neq 0, \\ &= 0 \qquad\quad \textit{if } x = 0 = \langle 0, 0, 0, \dots \rangle, \end{aligned}$$
$$\mathrm{Re}(y, x) = \langle y, x_1, x_2, \dots \rangle \quad \textit{where } x = \langle x_0, x_1, x_2, \dots \rangle.$$

PROOF. All of this is proven in Schwichtenberg [10], except the lower bound on $\langle x_0, \dots, x_n \rangle$:

$$\begin{aligned} \langle x_0, x_1, \dots, x_n \rangle &= t(b(x_n)0b(x_{n-1})0 \cdots 0b(x_0)) \\ &\geq \sum_{i=0}^{n} t(b(x_i))3^i \geq \sum_{i=0}^{n} 3^i x_i. \end{aligned}$$

In §3 we shall use the more common encoding $[a_0, \dots, a_n] = p_0^{a_0} p_1^{a_1} \dots p_n^{a_n}$, where $\{p_i\}_{i \in \mathfrak{N}}$ is the primitive recursive enumeration of the primes ($p_0 = 2$). The coordinate function for this encoding is

$$\mathrm{Co}(z, i) = (\mu a \leq z)[p_i^a | z \text{ and } p_i^{a+1} \nmid z],$$

which is primitive recursive.

In §4 we shall define a sequence of functions which proceeds by enumeration. The definition of this sequence depends on the assignation of an *index* $\#_h(f)$ to every function f in $\mathcal{E}(h)$, according to the following scheme:

$$\begin{array}{ll}
f(x_1, \dots, x_p) = h(x_1, \dots, x_p), & \#_h(h) = \langle 0, p \rangle, \\
f(x_1, \dots, x_n) = C_n^c(x_1, \dots, x_n) = c, & \#_h(C_n^c) = \langle 1, n, c \rangle, \\
f(x_1, \dots, x_n) = U_n^i(x_1, \dots, x_n) = x_i, & \#_h(U_n^i) = \langle 2, n, i \rangle, \\
f(x, y) = x + y, & \#_h(+) = \langle 3, 2 \rangle, \\
f(x, y) = x \dot{-} y, & \#_h(\dot{-}) = \langle 4, 2 \rangle, \\
f(x_1, \dots, x_n) & \#_h(f) = \langle 5, n, \#_h(g), \#_h(k_1), \\
\quad = g(k_1(x_1, \dots, x_n), \dots, k_m(x_1, \dots, x_n)), & \qquad \dots, \#_h(k_m) \rangle, \\
f(x_1, \dots, x_n) = \sum_{i < x_1} g(i, x_2, \dots, x_n), & \#_h(f) = \langle 6, n, \#_h(g) \rangle, \\
f(x_1, \dots, x_n) = \prod_{i < x_1} g(i, x_2, \dots, x_n), & \#_h(f) = \langle 7, n, \#_h(g) \rangle.
\end{array}$$

This scheme of assigning indices to functions of $\mathcal{E}(h)$ differs slightly but inconsequentially from that of Schwichtenberg [10]; the above scheme is closer to Grzegorczyk's definition of the class of elementary functions in [4].

Any function f which is elementary in h receives infinitely many indices under this scheme; the scheme actually assigns indices to the various ways of defining f in $\mathfrak{E}(h)$. However, this abuse of notation should not harm us.

If f is an elementary function, then f can be defined without recourse to the given initial function h. An index of an elementary function, with no instance of $\langle 0, p\rangle = \#_h(h)$ in it, is called an *absolute index*, denoted $\#(f)$.

Notice that the class of indices and the class of absolute indices are primitive recursive. Consequently there are primitive recursive predicates

Ind b iff b is an index of a function,

AInd b iff b is an absolute index of a function.

Now let

$$el^h(b, a) = \begin{cases} f((a)_0, (a)_1, \dots, (a)_{n-1}) & \text{if } b \text{ is an index of } f \text{ relative to } h, \\ 0 \quad \text{otherwise.} \end{cases}$$

Then $el^h \not\in \mathfrak{E}(h)$; however, el^h is primitive recursive in h.

The functions which we shall use to maniuplate this enumeration of $\mathfrak{E}(h)$ are listed in the next lemma. Parts (i) and (ii) are found in Schwichtenberg [10], and part (iii) is proven similarly.

LEMMA 2. *There are elementary functions Sb_n^m, It and Mn_n such that*:

(i) *If* $f(x_1, \dots, x_n) = g(k_1(x_1, \dots, x_n), \dots, k_m(x_1, \dots, x_n))$ *and* $f, g, k_1, \dots, k_m \in \mathfrak{E}(h)$, *then* $Sb_n^m(\#_h(g), \#_h(k_1), \dots, \#_h(k_m))$ *is an index of* f, *relative to* h.

(ii) *If* $f(x) = g^n(x)$, *where* $f, g \in \mathfrak{E}(h)$, *then* $It(n, \#_h(g))$ *is an index of* f, *relative to* h.

(iii) *If* $f(y, x_1, \dots, x_n) = (\mu z \leqslant y) Q(z, x_1, \dots, x_n)$, *where* Q *is a relation elementary in* h, *then* $Mn_n(\#_h(\mathrm{Char}_Q))$ *is an index of* f, *relative to* h.

A *normal function* is a continuous, strictly increasing function mapping an initial segment of the ordinals to ordinals. Normal functions are discussed in Veblen [13].

If λ is a countable limit ordinal, then a *fundamental sequence* for λ is a strictly increasing sequence of ordinals, whose supremum (or limit) is λ. In contexts where only one fundamental sequence for λ is under discussion, we write $\lambda[n]$ for the nth term in the fundamental sequence. Veblen [13] defined the following *standard fundamental sequences* for ordinals $< \varepsilon_0$. If $\alpha < \varepsilon_0$, then there exist unique $\gamma < \alpha$ and minimal $\beta < \alpha$ such that $\alpha = \beta + \omega^\gamma$. Further, if α is a limit, then $\gamma \neq 0$. So by transfinite recursion we can define

$$\alpha[n] = \begin{cases} \beta + \omega^{\gamma-1}n & \text{if } \gamma \text{ is a successor,} \\ \beta + \omega^{\gamma[n]} & \text{if } \gamma \text{ is a limit.} \end{cases}$$

This technique first breaks down at ε_0, since ε_0 is the least ordinal such that $\varepsilon_0 = \omega^{\varepsilon_0}$, by definition.

Perhaps a clearer, if less succinct, way to think of the standard fundamental sequences is as follows: Any ordinal $\alpha > 0$ can be expressed uniquely in the so-called Cantor normal form

$$\alpha = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_k}$$

where $\alpha \geqslant \alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_k$. Further, if α is not an ε-number (in particular, if $\alpha < \varepsilon_0$), then $\alpha > \alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_k$. From the definition of standard fundamental sequences, we have

$$\begin{aligned}
\alpha[n] &= ((\omega^{\alpha_1} + \cdots + \omega^{\alpha_{k-1}}) + \omega^{\alpha_k})[n] \\
&= (\omega^{\alpha_1} + \cdots + \omega^{\alpha_{k-1}}) + \omega^{\alpha_k}[n] \\
&= \begin{cases} \omega^{\alpha_1} + \cdots + \omega^{\alpha_{k-1}} + \omega^{\alpha_k - 1}n & \text{if } \alpha_k \text{ is a successor,} \\ \omega^{\alpha_1} + \cdots + \omega^{\alpha_{k-1}} + \omega^{\alpha_k[n]} & \text{if } \alpha_k \text{ is a limit.} \end{cases}
\end{aligned}$$

Using this formulation of the standard fundamental sequences, one readily proves:

LEMMA 3. *If* $\varepsilon_0 > \omega^{\alpha+1} > \beta$, *then* $(\omega^\alpha + \beta)[n] = \omega^\alpha + \beta[n]$.

3. **Systems of notation.** A *system of notation* consists of a set $\mathcal{S}$, a function $|\ |$ (called the *absolute value* function) mapping $\mathcal{S}$ onto an initial segment of the countable ordinals, and a function Seq: $\mathcal{S} \times \mathcal{N} \to \mathcal{S}$, such that these three conditions hold:

(i) if $|s| = 0$, then $\operatorname{Seq}(s, n) = s$ for all $n \in \mathcal{N}$;

(ii) if $|s|$ is a successor, then $\operatorname{Seq}(s, n) = \operatorname{Seq}(s, m)$ for all $n, m \in \mathcal{N}$, and $|\operatorname{Seq}(s, 0)| + 1 = |s|$;

(iii) if $|s|$ is a limit, then $|\operatorname{Seq}(s, n)|$, as a function of n, is a fundamental sequence for $|s|$.

The following predicates, Zero, Suc and Lim, are elementary in a total extension of Seq:

Zero s iff $\operatorname{Seq}(s, 0) = s$,

Suc s iff $\operatorname{Seq}(s, 0) = \operatorname{Seq}(s, 1) \neq s$,

Lim s iff $\operatorname{Seq}(s, 0) \neq \operatorname{Seq}(s, 1)$.

These predicates distinguish between notations whose absolute values are 0, a successor or a limit, respectively.

We will usually write $s[n]$ for $\operatorname{Seq}(s, n)$, even if not Lim s. Let $s[n_1, n_2] = (s[n_1])[n_2]$, and in general let

$$s[n_1, \ldots, n_k, n_{k+1}] = (s[n_1, \ldots, n_k])[n_{k+1}].$$

If Zero s or Suc s, we may write $P(s)$ for $\operatorname{Seq}(s, 0)$. If $n < \omega$ we will write $\bar{n}$ for any notation such that $|\bar{n}| = n$. The ambiguity in our notation will not harm us, since in our applications (§4) the functions indexed by finite notations $\bar{n}$ will be determined by the absolute value n.

Associated with a system of notation $\mathbb{S}$ is a partial ordering $<_{\mathbb{S}}$, defined as follows:

$s <_{\mathbb{S}} t$ iff there exist $n_1, \ldots, n_k$ $(k > 0)$ such that $s = t[n_1, \ldots, n_k]$.

This relation is antireflexive, since if $s <_{\mathbb{S}} t$, then $|s| < |t|$. It is transitive, since if $r = s[n_1, \ldots, n_k]$ and $s = t[m_1, \ldots, m_p]$ then

$$r = t[m_1, \ldots, m_p, n_1, \ldots, n_k].$$

So it is a partial order.

A *norm* on a system of notation $\mathbb{S}$ is a function $N: \mathbb{S} \to \mathfrak{N}$ such that the following three conditions hold:

(i) $N(s) = 0$ iff Zero s, for all $s \in \mathbb{S}$,

(ii) if $s \in \mathbb{S}$ and Suc s, then $N(P(s)) < N(s)$,

(iii) if $s \in \mathbb{S}$ and Lim s, then $N(s[n]) < N(s[n+1])$, for all $n \in \mathfrak{N}$.

In view of (ii), $N(\bar{k}) \geqslant k$, for all $k \in \mathfrak{N}$. Hence for a normed system of notation, the following predicates are primitive recursive in any total extensions of Seq and N:

Fin s iff for all $k \leqslant N(s)$, not Lim $s[0, \ldots, 0]$ (k zeroes) iff $|s| < \omega$.

Inf s iff not Fin s iff $|s| \geqslant \omega$.

The function

$\mathrm{Abs}(s) = (\mu z \leqslant N(s))$ [not Suc $P^z(s)$]

is also primitive recursive in any total extensions of Seq and N. Note that if $|s| = \lambda + n$, where λ is 0 or a limit, and $n < \omega$, then $\mathrm{Abs}(s) = n$. In particular, if Fin s, then $\mathrm{Abs}(s) = |s|$.

A normed system of notation is *regulated* if, for all $s, t \in \mathbb{S}$, if $|s| < |t|$ and Lim t, then $|s| \leqslant |t[N(s)]|$. This condition is used to insure that the fundamental sequences rise "fast enough" since

$$|s[n]| \geqslant \sup\{|t| \mid |t| < |s| \text{ and } N(t) \leqslant n\},$$

in a regulated system of notation. We shall make a more detailed analysis of the role of this condition after we have defined the subrecursive hierarchy equivalence property in the next section.

A normed system of notation is *p.r.-regulated* if it is regulated, and Seq and N can be extended to primitive recursive functions.

The remainder of this section will present a nontrivial p.r.-regulated system of notation, based on Schütte's Klammersymbols [9]. Because this system has an initial segment containing, for each limit $\alpha < \varepsilon_0$, a notation s such that $|s| = \alpha$ and $|s[n]|$ is the standard fundamental sequence $\alpha[n]$ of α, the subrecursive hierarchy defined on this system extends those previously in print. However, the results of §4 are true of all p.r.-regulated systems of notation, and are independent of the remainder of this section.

In Schütte's work a Klammersymbol is a symbol

$$\begin{pmatrix} a_0\, a_1 \cdots a_n \\ b_0\, b_1 \cdots b_n \end{pmatrix}$$

where $a_0, \ldots, a_n, b_0, \ldots, b_n$ are ordinals and $b_0 < b_1 < \cdots < b_n$. Schütte shows how to extend the domain of any normal function f such that $f(0) > 0$ to the class of Klammersymbols. We shall be utilizing Schütte's extension of the normal function $f(a) = 1 + a$. In that case the extension of f, as defined by Schütte, is as follows:

(i) $f\binom{a}{0} = f(a) = 1 + a$.

(ii)

$$f\begin{pmatrix} a_0 & a_1 \cdots a_n \\ 0 & b_1 \cdots b_n \end{pmatrix},$$

where $a_1 \neq 0$, is the a_0th common solution of the equations

$$x = f\begin{pmatrix} x & a_1' & a_2 \cdots a_n \\ b_0 & b_1 & b_2 \cdots b_n \end{pmatrix}$$

where $a_1' < a_1$ and $b_0 < b_1$.

(iii) If A and B are Klammersymbols which can be obtained from each other by adding or deleting columns of the form $\binom{0}{b}$, then $f(A) = f(B)$.

Schütte proves that this extension of f is well defined. As Feferman remarks in [3, p. 219], $f\binom{a\ b}{0\ 1} = \omega^b + a$, $f\binom{a\ b\ 1}{0\ 1\ 2} = \kappa^{(b)}(a)$ and $f\binom{a\ 2}{0\ 2} = \Gamma_a$.

Schütte proves the following useful facts:

LEMMA 4. (i)

$$f\begin{pmatrix} x\, a_1 \cdots a_n \\ b_0\, b_1 \cdots b_n \end{pmatrix}$$

is a normal function of x.

(ii)

$$a_i < f\begin{pmatrix} a_0\, a_1 \cdots a_n \\ b_0\, b_1 \cdots b_n \end{pmatrix} \quad \textit{if } a_0 \neq 0 \textit{ and } i > 0.$$

PROOF. (i) 3.2, 3.3, and 3.5 on p. 17 of [9].

(ii) 6.1 on p. 23 of [9]. □

We shall be dealing with notations for ordinals rather than ordinals themselves. We shall let the natural number 0 be the only notation for the ordinal 0. The other notations will be encodings of Klammersymbols. However, our Klammersymbols will have notations in the positions where Schütte has ordinals. We want a way to encode such Klammersymbols in the natural numbers. As a start, let us define

$$\left\langle \begin{matrix} a_0\, a_1 \cdots a_n \\ b_0\, b_1 \cdots b_n \end{matrix} \right\rangle = p_0^{a_0} p_1^{b_0} p_2^{a_1} p_3^{b_1} \cdots p_{2n}^{a_n} p_{2n+1}^{b_n}$$

$$= \prod_{i \leqslant n} p_{2i}^{a_i} p_{2i+1}^{b_i}$$

where $\{p_i\}$ is the primitive recursive sequence of primes.

This coding is not quite suitable, since it distinguishes matrices which differ only by the presence or absence of columns headed by 0, whereas Schütte's extension of f does not. By course-of-values recursion, we define the primitive recursive function C, to "compress" out the columns headed by zero:

$$C(0) = 0,$$

$$C(1) = 1,$$

$$C(z) = \begin{cases} C\left(\prod_{i<2n} p_i^{\mathrm{Co}(z,i)} \prod_{2n+1<i\leqslant z} p_i^{\mathrm{Co}(z,i+2)} \right) \\ \quad \text{where } n \text{ is the least } j \leqslant z \text{ such that} \\ \quad \mathrm{Co}(z, 2j) = 0 \text{ and there exists } m, \\ \quad 2j < m \leqslant z, \text{ such that } \mathrm{Co}(z, m) \neq 0, \\ \quad \text{if such } j \text{ exists;} \\ z \quad \text{otherwise.} \end{cases}$$

Now let

$$\begin{pmatrix} a_0 & a_1 \cdots a_n \\ b_0 & b_1 \cdots b_n \end{pmatrix} = C \left\langle \begin{matrix} a_0 & a_1 \cdots a_n \\ b_0 & b_1 \cdots b_n \end{matrix} \right\rangle.$$

This is the encoding of the Klammersymbols we shall use.

Now we can define the underlying set $\mathbb{S}$ and the absolute value function of our system of notation, as follows:

$0 \in \mathbb{S}$, $|0| = 0$; if $a_0, \ldots, a_n, b_0, \ldots, b_n \in \mathbb{S}$ and $|b_0| < \cdots < |b_n|$, then

$$\begin{pmatrix} a_0 \cdots a_n \\ b_0 \cdots b_n \end{pmatrix} \in \mathbb{S},$$

and

$$\begin{vmatrix} a_0 \cdots a_n \\ b_0 \cdots b_n \end{vmatrix} = f\begin{pmatrix} |a_0| \cdots |a_n| \\ |b_0| \cdots |b_n| \end{pmatrix}.$$

We adopt the following lexicographic ordering $<_1$ and lexicographic equivalence $=_1$ from Schütte: $\bar{0} =_1 \bar{0}$, and $\bar{0} <_1 t$ if $t \neq \bar{0}$; if

$$s = \begin{pmatrix} a_0 \cdots a_n \\ b_0 \cdots b_n \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} c_0 \cdots c_n \\ d_0 \cdots d_n \end{pmatrix},$$

where $|b_0| = |d_0| < |b_1| = |d_1| < \cdots < |b_n| = |d_n|$, then $s =_1 t$ if $|a_i| = |c_i|$ for all i, and $s <_1 t$ if there exists j such that $|a_j| < |c_j|$ but for all $k > j$, $|a_k| = |c_k|$.

Notice that in this definition, if either of the notations s or t lacks a column with a lower member whose absolute value is found in the other term, then the missing column can be supplied, with an upper member of $\bar{0}$. Hence, for all notations $s, t \in \mathbb{S}$, either $s <_1 t$, $s =_1 t$ or $s >_1 t$. The next lemma follows directly from the definition of $<_1$ and $=_1$.

LEMMA 5. *Suppose*

$$s = \begin{pmatrix} a_0 \cdots a_n \\ b_0 \cdots b_n \end{pmatrix} \quad \textit{and} \quad t = \begin{pmatrix} c_0 \cdots c_n \\ d_0 \cdots d_n \end{pmatrix},$$

where $|b_0| < |b_1|$, $|d_0| < |d_1|$, $|b_1| = |d_1| < \cdots < |b_n| = |d_n|$, *and* $|a_i| = |c_i|$ *for all* i, $2 \leqslant i \leqslant n$. *Suppose* $|a_1| \leqslant |c_1|$, $|b_0| \leqslant |d_0|$ *and* $|a_0| < |c_0|$. *Then* $s <_1 t$.

Schütte proves the following useful lemma [9, p. 25 in the middle]:

LEMMA 6. *If* $s =_1 t$, *then* $|s| = |t|$. *On the other hand, suppose*

$$s = \begin{pmatrix} a_0 \cdots a_n \\ b_0 \cdots b_n \end{pmatrix} <_1 t.$$

Then the following three statements determine the order of $|s|$ *and* $|t|$:

(i) $|s| < |t|$ *iff* $|a_i| < |t|$ *for all* i;

(ii) $|s| = |t|$ *iff there exists* i *such that* $|a_i| = |t|$, *and such that for all* j, *if* $j < i$ *then* $a_j = \bar{0}$, *and if* $j > i$ *then* $|a_j| < |t|$;

(iii) $|s| > |t|$ *otherwise, i.e., iff one of these two subcases hold*: (a) *there exists* j *such that* $|a_j| > |t|$; (b) *there exist* j, k *such that* $a_j \neq \bar{0}, j < k$ *and* $|a_k| = |t|$.

Our next objective is to define Seq. To do this we will need to be able to recognize which notations are zero, successor or limit notations, by means of a primitive recursive function K. (The predicates Zero, Suc and Lim are defined using Seq, so they are not yet available to us.) As a step towards distinguishing successor and limit notations, let us define the primitive recursive predicate

$$\text{Eps}\, s \quad \text{iff} \quad s \neq \begin{pmatrix} \text{Co}(s,0) & \text{Co}(s,2) \\ \bar{0} & \bar{1} \end{pmatrix}.$$

LEMMA 7. *If* $s \in \mathbb{S}$ *and* Eps s, *then* $|s|$ *is an epsilon number.*

Note. The converse is not true, for if $|s|$ is an epsilon number, then

$|{s \atop \bar{0}}| = 1 + |s| = |s|$ is also an epsilon number, but not Eps $({s \atop \bar{0}})$.

PROOF. If $s \in \mathbb{S}$ and Eps s, then

$$s = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ \bar{0} & \bar{1} & \cdots & b_n \end{pmatrix},$$

where $|b_n| > 1$ and $a_n \neq \bar{0}$. Hence

$$\begin{pmatrix} s \\ \bar{1} \end{pmatrix} = \begin{bmatrix} s & \bar{0} \\ \bar{1} & b_n \end{bmatrix} <_1 \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ \bar{0} & \bar{1} & \cdots & b_n \end{pmatrix} = s.$$

By Lemma 6(ii), $|{s \atop \bar{1}}| = |s|$. Hence $\omega^{|s|} = |s|$. Consequently $|s|$ is an epsilon number. $\square$

Of course, every epsilon number is a limit, so if Eps s then $|s|$ is a limit. On the other hand, if $s \neq \bar{0}$ and not Eps s, then $s = ({a \atop \bar{0}} {b \atop \bar{1}})$, and consequently $|s| = \omega^{|b|} + |a|$. Hence $|s|$ is a successor iff either $a = b = \bar{0}$ or $|a|$ is a successor; $|s|$ is a limit iff either $a = \bar{0} \neq b$, or $|a|$ is a limit. Accordingly we will define the function K by course-of-values recursion:

$$K(s) = \begin{cases} 0 & \text{if } s = \bar{0}, \\ 1 & \text{if } s = \bar{1} = \binom{\bar{0}}{\bar{0}}, \\ 2 & \text{if Eps } s\text{, or if } s = \binom{b}{\bar{1}} \text{ for some } b \neq \bar{0}, \\ K(a) & \text{if } s = \left({a \atop \bar{0}}\ {b \atop \bar{0}}\right) \text{ for some } a \neq \bar{0}. \end{cases}$$

K is primitive recursive, since the existential quantifiers in the third and fourth lines of the definition are bounded by s. Also, if $s \in \mathbb{S}$, then

$$K(s) = \begin{cases} 0 & \text{if } |s| = 0, \\ 1 & \text{if } |s| \text{ is a successor,} \\ 2 & \text{if } |s| \text{ is a limit.} \end{cases}$$

Now to define Seq. If $K(s) = 0$ then $s = \bar{0}$ and we must define $\mathrm{Seq}(s, n) = \bar{0}$ for all n. If $K(s) = 1$, then either $s = \bar{1}$ or $s = ({a \atop \bar{0}} {b \atop \bar{1}})$ where $K(a) = 1$. For these cases we define Seq by course-of-values recursion on s:

$$\mathrm{Seq}(s, n) = \begin{cases} \bar{0} \quad \text{if } s = \bar{1}, \\ \begin{pmatrix} \mathrm{Seq}(a,n) & b \\ \bar{0} & \bar{1} \end{pmatrix} \quad \text{if } s = \begin{pmatrix} a & b \\ \bar{0} & \bar{1} \end{pmatrix} \text{ for some } a, K(a) = 1. \end{cases}$$

One readily proves, by complete induction on s, that if $K(s) = 1$, then $|\mathrm{Seq}(s, n)| + 1 = |s|$, for all $n \in \mathfrak{N}$.

If $K(s) = 2$, we will distinguish three types of limit notations. Let

$$s = \begin{pmatrix} a_0 \cdots a_m \\ b_0 \cdots b_m \end{pmatrix} \quad \text{where } a_i \neq \bar{0} \text{ for all } i, 0 < i < m.$$

Then

Type$_1$ s iff $K(a_0) = 2$;
Type$_2$ s iff $K(a_0) = 1$ and $b_0 = \bar{0}$;
Type$_3$ s iff $K(a_0) = 1$ and $b_0 \neq \bar{0}$.
If Type$_1$ s, then let

$$\mathrm{Seq}(s, n) = \begin{pmatrix} \mathrm{Seq}(a_0, n) & a_1 \cdots a_m \\ b_0 & b_1 \cdots b_m \end{pmatrix}.$$

If Type$_2$ s, then Seq(s, n) is defined by primitive recursion on n:

$$\mathrm{Seq}(s, 0) = \mathrm{Sc}\begin{bmatrix} \mathrm{Seq}(a_0, 0) & a_1 \cdots a_m \\ \bar{0} & b_1 \cdots b_m \end{bmatrix},$$

$$\mathrm{Seq}(s, n+1) = \begin{pmatrix} \mathrm{Seq}(s, n) & \mathrm{Seq}(a_1, n) & a_2 \cdots a_m \\ \mathrm{Seq}(b_1, n) & b_1 & b_2 \cdots b_m \end{pmatrix},$$

where $\mathrm{Sc}(t) = (\begin{smallmatrix} \bar{1} & t \\ \bar{0} & \bar{1} \end{smallmatrix})$. We shall observe at this point that if Type$_2$ s, then Eps s. For if not Eps s, then $s = (\begin{smallmatrix} a & b \\ \bar{0} & \bar{1} \end{smallmatrix})$, where $a = a_0$, $b = a_1$. But then $|s| = \omega^{|b|} + |a|$, which is a successor, since $K(a) = K(a_0) = 1$ when Type$_2$ s. This contradicts the assumption that $K(s) = 2$. Consequently, Eps s if Type$_2$ s. Then we also have Eps t, where

$$t = \begin{bmatrix} \mathrm{Seq}(a_0, 0) & a_1 \cdots a_m \\ \bar{0} & b_1 \cdots b_m \end{bmatrix}.$$

Hence

$$|s[0]| = \begin{vmatrix} \bar{1} & t \\ \bar{0} & \bar{1} \end{vmatrix} = \omega^{|t|} + 1 = |t| + 1.$$

If Type$_3$ s, then Seq(s, n) is given by primitive recursion on n:

$$\mathrm{Seq}(s, 0) = \bar{0},$$

$$\mathrm{Seq}(s, n+1) = \begin{pmatrix} \mathrm{Seq}(s, n) & \mathrm{Seq}(a_0, 0) & a_1 \cdots a_m \\ \mathrm{Seq}(b_0, n) & b_0 & b_1 \cdots b_m \end{pmatrix}.$$

It is easy to prove, by complete induction on s, that if $K(s) = 2$, then $\{|\mathrm{Seq}(s, n)|\}_{n<\omega}$ is a fundamental sequence for $|s|$.

The definition of Seq, which we have broken up in pieces, can be collected into a single definition, proceeding by course-of-values recursion on s and primitive recursion on n. This is an unnested double recursion, and hence is reducible to a primitive recursion.

The norm we choose to regulate $\mathcal{S}$ is a "symbol counting" norm:

$$N(\bar{0}) = 0,$$

$$N(\bar{1}) = N\begin{pmatrix}\bar{0}\\ \bar{0}\end{pmatrix} = 1,$$

$$N\begin{pmatrix}a_0 \cdots a_n\\ b_0 \cdots b_n\end{pmatrix} = 1 + N(a_0) + \cdots + N(a_n) + N(b_0) + \cdots + N(b_n),$$

$$\text{if } a_i \neq \bar{0} \text{ for all } i, 0 \leqslant i \leqslant n.$$

N is defined by course-of-values recursion, so it is primitive recursive. By complete induction on s, one can show that if $K(s) = 1$, then $N(P(s)) = N(s) - 1$; and if $K(s) = 2$, then $N(s[n]) < N(s[n+1])$ for all $n \in \mathfrak{N}$.

THEOREM 8. *If $s, t \in \mathcal{S}$, $|t|$ is a limit, and $|s| < |t|$, then $|s| \leqslant |t[N(s)]|$. Hence $\mathcal{S}$ is a p.r.-regulated system of notation.*

PROOF. If $s = \bar{0}$ then the theorem is trivial, so we may assume $s \neq \bar{0}$. The proof is by complete induction on t. There are two major cases: either $s <_1 t$ or $s >_1 t$.

Case (i): $s <_1 t$. We can write

$$s = \begin{pmatrix}a_0 \cdots a_i \cdots a_{i+m}\\ b_0 \cdots b_i \cdots b_{i+m}\end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix}c_0 \cdots c_j \cdots c_{j+m}\\ d_0 \cdots d_j \cdots d_{j+m}\end{pmatrix},$$

where $|b_{i+k}| = |d_{j+k}|$ for all k, $0 \leqslant k \leqslant m$; $|a_i| < |c_j|$; $|a_{i+k}| = |c_{j+k}|$ for all k, $1 \leqslant k \leqslant m$; $a_k \neq \bar{0}$ for all $k < i$; and $c_k \neq \bar{0}$ for all $k < j$. By Lemma 4, $|a_k| \leqslant |s| < |t|$ for all k, $0 \leqslant k \leqslant i + m$. If we can also show that $s \leqslant_1 t[N(s)]$, then we can deduce that $|s| \leqslant |t[N(s)]|$, by Lemma 6. We shall either do this, or show $|s| \leqslant |t[N(s)]|$ directly, for each of the following seven subcases:

(a) $j \geqslant 2$,
(b) $j = 1$, $\text{Type}_1\ t$,
(c) $j = 1$, $\text{Type}_2\ t$,
(d) $j = 1$, $\text{Type}_3\ t$,
(e) $j = 0$, $\text{Type}_1\ t$,
(f) $j = 0$, $\text{Type}_2\ t$,
(g) $j = 0$, $\text{Type}_3\ t$.

Subcase (i, a): $j \geqslant 2$. Then $t[N(s)]$ is of the form

$$\begin{pmatrix}c'_0 \cdots c'_{j-1} & c_j \cdots c_{j+m}\\ d'_0 \cdots d'_{j-1} & d_j \cdots d_{j+m}\end{pmatrix}$$

so $s <_1 t[N(s)]$, for the same reason $s <_1 t$, since the decisive columns (j through $j+m$) for the lexicographical ordering are the same for t and $t[N(s)]$.

Subcase (i, b): $j = 1$, $\text{Type}_1\ t$. Then

$$t[N(s)] = \begin{pmatrix} c_0[N(s)] & c_1 \cdots c_{1+m} \\ d_0 & d_1 \cdots d_{1+m} \end{pmatrix}.$$

Again the decisive columns for the lexicographical ordering relative to s are the same for t and $t[N(s)]$, so $s <_1 t[N(s)]$.

Subcase (i, c): $j = 1$, $\text{Type}_2\ t$. Then

$$t[N(s)] = \begin{bmatrix} t[N(s)-1] & c_1[N(s)-1] & c_2 \cdots c_{1+m} \\ d_1[N(s)-1] & d_1 & d_2 \cdots d_{1+m} \end{bmatrix}.$$

If Lim c_1, then by induction

$$|a_i| \leqslant |c_1[N(a_i)]| < |c_1[N(a_i) + N(b_i)]| \leqslant |c_1[N(s)-1]|,$$

so $s <_1 t$. If Suc c_1 then $|a_i| \leqslant |P(c_1)| = |c_1[N(s)-1]|$. If $|a_i| < |P(c_1)|$ then $s <_1 t[N(s)]$. If $|a_i| = |P(c_1)|$, we reason: If $i = 0$, then the extra leading column in $t[N(s)-1]$ gives $s <_1 t[N(s)]$; if $i > 0$ then $|b_{i-1}| < |b_i| = |d_1|$, so by induction $|b_{i-1}| \leqslant |d_1[N(b_{i-1})]| < |d_1[N(s)-1]|$, whence $s <_1 t[N(s)]$.

Subcase (i, d): $j = 1$, $\text{Type}_3\ t$. Then

$$t[N(s)] = \begin{bmatrix} t[N(s)-1] & P(c_0) & c_1 \cdots c_{1+m} \\ d_0[N(s)-1] & d_0 & d_1 \cdots d_{1+m} \end{bmatrix}.$$

In this case the decisive columns of $t[N(s)]$ and t, for determining the lexicographical order relative to s, are the same. Hence $s <_1 t[N(s)]$.

Subcase (i, e): $j = 0$, $\text{Type}_1\ t$. Then

$$t[N(s)] = \begin{pmatrix} c_0[N(s)] & c_1 \cdots c_m \\ d_0 & d_1 \cdots d_m \end{pmatrix}.$$

By transfinite induction,

$$|a_i| \leqslant |c_0[N(a_i)]| < |c_0[N(a_i)+1]| \leqslant |c_0[N(s)]|.$$

Consequently, $s <_1 t[N(s)]$.

Subcase (i, f): $j = 0$, $\text{Type}_2\ t$. Then $|b_i| = |d_0| = |\bar{0}| = 0$, so $b_i = \bar{0}$, whence $i = 0$. Thus

$$s = \begin{pmatrix} a_0 & a_1 \cdots a_m \\ \bar{0} & b_1 \cdots b_m \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} c_0 & c_1 \cdots c_m \\ \bar{0} & d_1 \cdots d_m \end{pmatrix},$$

where $|a_0| < |c_0|$. By Lemma 4 (i),

$$|s| \leqslant \begin{vmatrix} P(c_0) & c_1 \cdots c_n \\ \bar{0} & d_1 \cdots d_n \end{vmatrix} < |t[0]| < |t[N(s)]|.$$

Subcase (i, g): $j = 0$, Type$_3$ t. Then

$$t[N(s)] = \begin{bmatrix} t[N(s)-1] & P(c_0) & c_1 \cdots c_m \\ d_0[N(s)-1] & d_0 & d_1 \cdots d_m \end{bmatrix}.$$

By transfinite induction, $|a_{i-1}| \leqslant |t[N(a_{i-1})]| \leqslant |t[N(s)-1]|$, and similarly $|b_{i-1}| \leqslant |d_0[N(s)-1]|$. Also $|a_i| \leqslant |P(c_0)|$, since $|a_i| < |c_0|$. By Lemma 5, $s <_1 t[N(s)]$.

This concludes the proof of Case (i).

As for Case (ii), $s >_1 t$, let us write

$$t = \begin{pmatrix} c_0 \cdots c_m \\ d_0 \cdots d_m \end{pmatrix}$$

where $c_i \neq \bar{0}$ for all i, $0 \leqslant i \leqslant m$. By Lemma 6(iii),

either $|s| < |c_k|$ for some k, $0 \leqslant k \leqslant m$,

or $|s| = |c_k|$ for some k, $1 \leqslant k \leqslant m$.

It is more convenient to consider the equivalent pair of subcases:

either $|s| < |c_0|$

or $|s| \leqslant |c_k|$ for some k, $1 \leqslant k \leqslant m$.

Each of these possibilities will be further analyzed according to the type of t, giving six subcases in all.

Subcase (ii, a): $|s| < |c_0|$, Type$_1$ t. By transfinite induction,

$$|s| \leqslant |c_0[N(s)]| \leqslant \begin{vmatrix} c_0[N(s)] & c_1 \cdots c_{j+m} \\ d_0 & d_1 \cdots d_{j+m} \end{vmatrix} = |t[N(s)]|.$$

Subcase (ii, b): $|s| < |c_0|$, Type$_2$ t. Then

$$|s| \leqslant |P(c_0)| \leqslant \begin{vmatrix} P(c_0) & c_1 \cdots c_{j+m} \\ d_0 & d_1 \cdots d_{j+m} \end{vmatrix} < |t[0]| < |t[N(s)]|.$$

Subcase (ii, c): $|s| < |c_0|$, Type$_3$ t. Then

$$|s| \leqslant |P(c_0)| \leqslant \begin{vmatrix} \bar{0} & P(c_0) & c_1 \cdots c_m \\ d_0[0] & d_0 & d_1 \cdots d_m \end{vmatrix} = |t[1]| \leqslant |t[N(s)]|.$$

Subcase (ii, d): $|s| \leqslant |c_k|$ for some k, $1 \leqslant k \leqslant m$; Type$_1$ t. Then

$$|s| \leqslant |c_k| < \begin{vmatrix} c_0[0] & c_1 \cdots c_m \\ d_0 & d_1 \cdots d_m \end{vmatrix} = |t[0]| < |t[N(s)]|.$$

Subcase (ii, e): $|s| \leqslant |c_k|$ for some k, $1 \leqslant k \leqslant m$; Type$_2$ t. Then

$$|s| < |c_k| < \begin{vmatrix} P(c_0) & c_1 \cdots c_m \\ \bar{0} & d_1 \cdots d_m \end{vmatrix} < |t[0]| < |t[N(s)]|.$$

Subcase (ii, f): $|s| < |c_k|$ for some k, $1 < k < m$; $\text{Type}_3\ t$. Then

$$|s| < |c_k| < \begin{vmatrix} \bar{0} & P(c_0) & c_1 \cdots c_m \\ b_0[0] & d_0 & d_1 \cdots d_m \end{vmatrix} = |t[1]| < |t[N(s)]|.$$

This concludes the proof of the theorem. □

Finally we will show that $\mathbb{S}$ contains a set of notations corresponding to the standard fundamental sequences. The simplest notation for ε_0 is evidently $(\frac{\bar{1}}{\bar{2}})$. We shall see that if $s <_{\mathbb{S}} (\frac{\bar{1}}{\bar{2}})$ and Lim s, then $|s[n]| = |s|[n]$, where $|s|[n]$ is the standard fundamental sequence for $|s|$.

THEOREM 9. (i) *If* $s \in \mathbb{S}$, $s \neq \bar{0}$, *and* $s <_{\mathbb{S}} (\frac{\bar{1}}{\bar{2}})$, *then* $s = (\frac{a}{\bar{0}}\ \frac{b}{\bar{1}})$, *where* $|a| < \omega^{|b|+1}$.

(ii) *If* $s \in \mathbb{S}$, $s <_{\mathbb{S}} (\frac{\bar{1}}{\bar{2}})$ *and* Lim s, *then* $|s[n]| = |s|[n]$, *where* $|s|[n]$ *is the standard fundamental sequence for* $|s|$.

PROOF. (i) If $s <_{\mathbb{S}} (\frac{\bar{1}}{\bar{2}}) = e$, then $s = e[n_1, \ldots, n_k]$ for some natural numbers $n_1, \ldots, n_k$. The proof is by induction on k. If $k = 1$, then $s = e[n]$. The notation e is type 3, and its fundamental sequence is given by

$$e[0] = \bar{0}, \qquad e[n+1] = \begin{pmatrix} e[n] & \bar{0} \\ \bar{1} & \bar{2} \end{pmatrix} = \begin{bmatrix} \bar{0} & e[n] \\ \bar{0} & \bar{1} \end{bmatrix}.$$

Trivially $|\bar{0}| < \omega^{|e[n]|+1}$, so each of the nonzero terms of this sequence satisfies the result.

To prove the induction step, suppose $s = e[n_1, \ldots, n_k, n] = t[n]$ where $t = e[n_1, \ldots, n_k]$. By induction, either Zero t or $t = (\frac{a}{\bar{0}}\ \frac{b}{\bar{1}})$ where $|a| < \omega^{|b|+1}$. If Zero t, then Zero s, and there is nothing to prove. If Suc t, then $s = t[n] = P(t) = (\frac{P(a)}{\bar{0}}\ \frac{b}{\bar{1}})$, where $|P(a)| < |a| < \omega^{|b|+1}$. If Lim t, there are three cases: Lim a; or $a = \bar{0}$ and Lim b; or $a = \bar{0}$ and Suc b.

Case (i, a): Lim a. Then $\text{Type}_1\ t$, and $t[n] = (\frac{a[n]}{\bar{0}}\ \frac{b}{\bar{1}})$, where $|a[n]| < |a| < \omega^{|b|+1}$.

Case (i, b): $a = \bar{0}$ and Lim b. Then $\text{Type}_1 t$, and

$$t[n] = \begin{pmatrix} b[n] \\ \bar{1} \end{pmatrix} = \begin{bmatrix} \bar{0} & b[n] \\ \bar{0} & \bar{1} \end{bmatrix}$$

where $|\bar{0}| < \omega^{|b[n]|+1}$.

Case (i, c): $a = \bar{0}$ and Suc b. Then $\text{Type}_3\ t$, and

$$t[0] = \bar{0}, \qquad t[n+1] = \begin{pmatrix} t[n] & P(b) \\ \bar{0} & \bar{1} \end{pmatrix}.$$

Induction on n shows that $|t[n]| = \omega^{|P(b)|}\, n$. For the nonzero terms $t[n+1]$ of this sequence, we have

$$|t[n]| = \omega^{|P(b)|}n < \omega^{|b|} = \omega^{|P(b)|+1},$$

again confirming the result.

(ii) By complete induction on $|s|$. In part (i), we have seen that $s = \binom{a\ b}{\bar{0}\ \bar{1}}$, where $|a| < \omega^{|b|+1}$. There are three cases: Lim a; or $a = \bar{0}$ and Lim b; or $a = \bar{0}$ and Suc b.

Case (ii, a): Lim a. Then Type$_1$ s, and

$$|s[n]| = \begin{vmatrix} a[n] & b \\ \bar{0} & \bar{1} \end{vmatrix} = \omega^{|b|} + |a[n]| = \left(\omega^{|b|} + |a|\right)[n] = |s|[n],$$

using Lemma 2 for the third equality.

Case (ii, b): $a = \bar{0}$ and Lim b. Then Type$_1$ s, and

$$|s[n]| = \begin{vmatrix} b[n] \\ \bar{1} \end{vmatrix} = \omega^{|b[n]|} = \omega^{|b|}[n] = |s|[n].$$

Case (ii, c): $a = \bar{0}$ and Suc b. Then Type$_3$ s, and

$$s[0] = \bar{0}, \qquad s[n+1] = \begin{pmatrix} s[n] & P(b) \\ \bar{0} & \bar{1} \end{pmatrix}.$$

By induction on n, $|s[n]| = \omega^{|P(b)|}n = \omega^{|b|}[n] = |s|[n]$.

4. **The subrecursive hierarchy equivalence property.** Given a system of notation $\mathfrak{S}$, we can define sequences of functions indexed by notations of the system. For example, in Schwichtenberg [10] we find

$$G_s(x) = \begin{cases} 2^x & \text{if Zero } s, \\ G^x_{P(s)}(x) & \text{if Suc } s, \\ G_{s[x]}(x) & \text{if Lim } s. \end{cases}$$

Let $\mathcal{G}_s = \mathfrak{E}(G_s)$, and let $\mathcal{G}^*_s = \mathfrak{E}\{G_t \mid t \leqslant_{\mathfrak{S}} s\}$.

Unfortunately, for arbitrary systems of notation, we cannot prove even simple desirable properties of the functions G_s, such as strict monotonicity. To get around this, Löb and Wainer [7] propose

$$F_s(x) = \begin{cases} 2^x & \text{if Zero } s, \\ F^x_{P(s)}(x) & \text{if Suc } s, \\ F_{s[x]}(\rho_s(x)) & \text{if Lim } s, \end{cases}$$

where $\rho_s(0) = 0$, $\rho_s(1) = 1$, $\rho_s(x) = (\mu z > \rho_s(x-1))(\forall y < x)\ [F_{s[y]}(z) < F_{s[x]}(z)]$ if $x \geqslant 2$. Löb and Wainer show that F_s is totally defined for all $s \in \mathfrak{S}$ (this amounts to showing that ρ_s is totally defined for all limit notations), and that F_s is strictly increasing for all $s \in \mathbb{S}$. Let $\mathcal{F}_s = \mathcal{E}(F_s)$, and let $\mathcal{F}_s^* = \mathcal{E}\{F_t | t <_{\mathbb{S}} s\}$.

In addition to the functions G_s and F_s, we shall consider enumeration functions E_s, proposed by Schwichtenberg [10]. For $s \in \mathbb{S}$, let

$$E_s(b, a) = \begin{cases} 0 & \text{if Zero } s, \\ el^H(b, a) & \text{if Suc } s, \text{ where } H = E_{P(s)}, \\ el^H((b)_1, a) & \text{if Lim } s, \text{ where } H = E_{s[(b)_0]}. \end{cases}$$

Let $\mathcal{E}_s = \mathcal{E}(E_s)$. (There is no need to define $\mathcal{E}_s^*$, since $E_t \in \mathcal{E}_s$ if $t <_{\mathbb{S}} s$.) If $f \in \mathcal{E}_s$, let $\#_s(f) = \#_{E_s}(f)$.

We shall say that a system of notation has the *subrecursive hierarchy equivalence property* if it has the following two properties:

(i) $\mathcal{E}_s = \mathcal{F}_s = \mathcal{F}_s^* = \mathcal{G}_s = \mathcal{G}_s^*$ for all $s \in \mathbb{S}$;

(ii) if $|s| = |t|$ then $\mathcal{E}_s = \mathcal{E}_t$, for all $s, t \in \mathbb{S}$.

(Axt [1] called (ii) the uniqueness property.) Our objective is to prove that if $\mathbb{S}$ is a p.r.-regulated system of notation, then $\mathbb{S}$ has the subrecursive hierarchy equivalence property.

Although we shall not explicitly prove it, our techniques can be used to show that if Seq has a total extension which is primitive recursive, then $\mathcal{G}_s \subseteq \mathcal{G}_s^* \subseteq \mathcal{E}_s \subseteq \mathcal{F}_s \subseteq \mathcal{F}_s^*$ for all $s \in \mathbb{S}$. P.r.-regulation makes two contributions: first, it enables us to bound ρ_t by a primitive recursive function, uniformly in t (Lemma 12(i)); and second, it enables us to bound F_s in terms of G_t, if $|s| = |t|$ (Lemma 14). Because ρ_t is bounded by a primitive recursive function, uniformly in t, a recursion theorem argument found in Schwichtenberg [10] can be adapted to prove not just that $\mathcal{G}_s^* \subseteq \mathcal{E}_s$, but also $\mathcal{F}_s^* \subseteq \mathcal{E}_s$ (Theorem 21). Because F_s can be bounded in terms of G_t if $|s| = |t|$, we can adapt the proof that $\mathcal{E}_s \subseteq \mathcal{F}_s$ to also show that $\mathcal{E}_s \subseteq \mathcal{G}_t$ (Theorem 18). These strengthened containments suffice to prove the subrecursive hierarchy equivalence property.

The simple majorization properties of the functions F_s which we shall use are listed in the following lemma.

LEMMA 10. (i) $2x \leqslant G_s(x) \leqslant F_s(x)$ *for all* $s \in \mathbb{S}$.

(ii) $x^2 \leqslant F_{\bar{0}}(x) = G_{\bar{0}}(x) = 2^x$ *if* $x \neq 3$.

(iii) $F_{s[b]}(x) < F_s(x)$ *if* $b < x$, $2 \leqslant x$, *and not* Zero s.

(iv) $F_{\bar{0}}(x^y) \geqslant x^{y+1}$ *if* $x \geqslant 4$ *and* $y \geqslant 1$.

(v) $F_s(x) \geqslant G_s(x) \geqslant G_{\bar{k}}(x) = F_{\bar{k}}(x)$ *if* $x \geqslant \max(k, 1)$ *and* $|s| \geqslant |\bar{k}| = k$.

(vi) $F_s^m(x) \geqslant (F_s(x))^m$ *if* $x \geqslant 1$ *or* $m \geqslant 1$.

(vii) $F_s^n(x) + F_s^m(x) \leqslant F_s^{\max(m,n)+1}(x)$.

(viii) $F_s^n(x) F_s^m(x) \leqslant F_s^{\max(m,n)+1}(x)$.

(ix) $(F_s^n(x))^x \leqslant F_s^{n+2}(x)$.

PROOF. These are proven by either finite or transfinite induction. As an example, we prove (ix). The proof is by induction on n. If $n = 0$ and $x \neq 3$ then

$$\left(F_s^0(x)\right)^x = x^x \leqslant (2^x)^x = 2^{x^2} \leqslant 2^{2^x} = F_{\bar{0}}^2(x) \leqslant F_s^2(x).$$

If $n = 0$ and $x = 3$ then

$$\left(F_s^0(3)\right)^3 = 3^3 < 2^{2^3} = F_{\bar{0}}^2(3) \leqslant F_s^2(3).$$

Now the induction step: let $y = F_s(x)$; then

$$\left(F_s^{n+1}(x)\right)^x = \left(F_s^n F_s(x)\right)^x = \left(F_s^n(y)\right)^x \leqslant \left(F_s^n(y)\right)^y$$
$$\leqslant F_s^{n+2}(y) = F_s^{n+2}F_s(x) = F_s^{n+3}(x).$$

The following lemma is well known. Part (i) is proven in Löb and Wainer [7], and part (ii) is an immediate consequence, because of Lemma 10(v).

LEMMA 11. (i) *For all* $n \in \mathfrak{N}$, $\mathfrak{F}_{\bar{n}} = \mathcal{G}_{\bar{n}} = \mathcal{E}^{n+3}$.
(ii) *If* $\mathrm{Inf}\ s$, *then* $\mathcal{P} \subseteq \mathcal{G}_s$ *and* $\mathcal{P} \subseteq \mathfrak{F}_s$.

Let $M(s) = \max\{N(s), P^m(s) | m \leqslant \mathrm{Abs}(s)\}$. Then $M(s)$ is primitive recursive. Consequently $M(\mathrm{Seq}(s, x))$ is also primitive recursive. So there exists k such that if $\max(s, x) > k$, then $M(\mathrm{Seq}(s, x)) < G_t(\max(s, x))$ for all notations t such that $|t| \geqslant k$. Now if $x > \max(s, k)$, then $\max(s, x) = x > k$, so

$$M(\mathrm{Seq}(s, x)) < G_t(\max(s, x)) = G_t(x), \quad \text{for all } t,\ |t| \geqslant k.$$

The value of k will remain fixed through the rest of this paper. The next four lemmas (12–15) are the essential ones in the proof of the theorems in this section.

LEMMA 12. (i) *If* $\mathrm{Lim}\ t$, *then* $\rho_t(x) \leqslant G_{\bar{k}}(\max(t, x))$.
(ii) *If* $|s| < |t|$ *and* $x > \max(k, M(s))$, *then* $F_s(x) < G_t(x) \leqslant F_t(x)$.

PROOF. By simultaneous transfinite induction on $|t|$. If Zero t, then (i) and (ii) are vacuous.

Suppose Suc t. Then again (i) is vacuous. The proof of (ii) has four subcases: (a) $|s| < |P(t)|$; (b) $|s| = |P(t)| = 0$; (c) $|s| = |P(t)|$ and Suc s; (d) $|s| = |P(t)|$ and Lim s.

Subcase (a): Suc t and $|s| < |P(t)|$. Then by induction

$$F_s(x) < G_{P(t)}(x) \leqslant G_{P(t)}^x(x) = G_t(x).$$

Note that $x > \max(M(s), k)$, so $x \geqslant 1$. Also $G_{P(t)}(x) \geqslant x$, so $G_{P(t)}^x(x) \geqslant G_{P(t)}(x)$.

Subcase (b): Suc t, $|s| = |P(t)| = 0$. This is trivial.

Subcase (c): Suc t, $|s| = |P(t)|$ and Suc s. The transfinite induction hypothesis tells us

$$F_{P(s)}(x) < G_{P(t)}(x) \quad \text{if } x > \max(M(s), k) \geqslant \max\big(M(P(s)), k\big).$$

Finite iteration of this gives

$$F^n_{P(s)}(x) < G^n_{P(t)}(x) \quad \text{if } x > \max(M(s), k), \text{ for all } n > 0.$$

In particular, $F_s(x) = F^x_{P(s)}(x) < G^x_{P(t)}(x) = G_t(x)$.

Subcase (d): Suc t, $|s| = |P(t)|$ and Lim s. Since $|s| < |t|$, by complete induction using (i) we have

$$\rho_s(x) \leqslant G_{\bar{k}}(\max(s, x)) = G_{\bar{k}}(x)$$

since $x > k$ and $x > M(s) \geqslant s$. Hence

$$F_s(x) = F_{s[x]}\rho_s(x) \leqslant F_{s[x]}G_{\bar{k}}(x).$$

Now let $y = G_{P(t)}(x) \geqslant G_{\bar{k}}(x)$, since Inf $P(t)$ and $x > k$. So

$$F_s(x) \leqslant F_{s[x]}G_{\bar{k}}(x) \leqslant F_{s[x]}(y).$$

Now $y = G_{P(t)}(x) \geqslant G_{\bar{k}}(x) \geqslant M(s[x]) \geqslant N(s[x])$, so $|s[x]| < |P(t)[y]|$. Hence

$$F_s(x) \leqslant F_{s[x]}(y) \leqslant G_{P(t)[y]}(y) = G_{P(t)}(y)$$
$$= G_{P(t)}G_{P(t)}(x) = G^2_{P(t)}(x) \leqslant G^x_{P(t)}(x) = G_t(x).$$

Here note that $y > \max(M(s[x]), k)$, so that (ii) applies by induction. Also $x > \max(M(s), k) \geqslant M(s) \geqslant N(s) \geqslant 1$, so $x \geqslant 2$.

This completes the proof of (ii) if Suc t.

Suppose Lim t. With regard to (i),

$$\rho_t(0) = 0 \leqslant G_{\bar{k}}(\max(t, 0)), \qquad \rho_t(1) = 1 < G_{\bar{k}}(\max(t, 1)),$$

and for $x \geqslant 2$, if $y < x$ then by transfinite induction using (ii)

$$F_{t[y]}G_{\bar{k}}(\max(t, x)) < F_{t[x]}G_{\bar{k}}(\max(t, x)),$$

since $G_{\bar{k}}(\max(t, x)) > k$ when $x \geqslant 2$, and $G_{\bar{k}}(\max(t, x)) \geqslant G_{\bar{k}}(\max(t, y)) > M(t[y])$. By the definition of $\rho_t(x)$, $\rho_t(x) \leqslant G_{\bar{k}}(\max(t, x))$.

To prove (ii) if Lim t: if $|s| < |t|$ and $x > \max(k, M(s)) \geqslant N(s)$, then $|s| < |t[x]|$, so

$$F_s(x) < G_{t[x]}(x) = G_t(x) \leqslant F_t(x),$$

by complete induction.

LEMMA 13. *If* Lim t, $|s| < |t|$ *and* $x >$ $\max(k, M(s) + 1)$, *then* $F^x_s(x) < G_t(x) \leqslant F_t(x)$.

PROOF. We adjoin to the system of notation a notation r such that Suc r, $P(r) = s$, and $N(r) = N(s) + 1$. The resulting system is still regulated, for if Lim q and $|q| > |r| > |s|$, then $|q[N(r)]| = |q[N(s) + 1]| > |q[N(s)]| \geqslant |s|$, so $|q[N(r)]| \geqslant |r|$. So the expanded system of notation is p.r.-regulated. Now note that $M(r) \leqslant M(s) + 1$. So if $x > \max(k, M(s) + 1)$ then $x > \max(k, M(r))$. By the previous lemma, $F_s^x(x) = F_r(x) < G_t(x) \leqslant F_t(x)$ if $x > \max(k, M(s) + 1)$.

LEMMA 14. *If* $|s| = |t|$ *and* $y \geqslant G_k^-(\max(M(s), x))$, *then* $F_s(x) \leqslant G_t(y)$.

PROOF. By transfinite induction on $|t|$. If Fin t, then $F_s = G_t$, and the result is trivial since F_s is strictly increasing.

If Suc t and Inf t, we begin by proving

$$F_{P(s)}^n(x) \leqslant G_{P(t)}^{2n-1}(y) \quad \text{if } y \geqslant G_k^-\big(\max(M(s), x)\big)$$

for all positive integers n. The proof is by induction on n. If $n = 1$ and $y \geqslant G_k^-(\max(M(s), x)) \geqslant G_k^-(\max(M(P(s)), x))$, then by the transfinite induction hypothesis, $F_{P(s)}(x) \leqslant G_{P(t)}(y)$. For the finite induction step,

$$\begin{aligned} F_{P(s)}^{n+1}(x) = F_{P(s)}F_{P(s)}^n(x) &\leqslant F_{P(s)}G_{P(t)}^{2n-1}(y) \\ &\leqslant G_{P(t)}G_{P(t)}G_{P(t)}^{2n-1}(y) = G_{P(t)}^{2n+1}(y). \end{aligned}$$

As justification, note that

$$G_{P(t)}G_{P(t)}^{2n-1}(y) \geqslant G_k^-\Big(\max\big(M(s), G_{P(t)}^{2n-1}(y)\big)\Big)$$

since Inf $P(t)$. The second inequality now follows by the transfinite induction hypothesis.

To prove the lemma when Suc t, we have

$$F_s(x) = F_{P(s)}^x(x) \leqslant G_{P(t)}^{2x-1}(y) \leqslant G_{P(t)}^y(y) = G_t(y),$$

since $y \geqslant G_k^-(\max(M(s), x)) \geqslant 2x - 1$.

To prove the lemma if Lim t, we have

$$\begin{aligned} F_s(x) = F_{s[x]}\rho_s(x) &\leqslant F_{s[x]}G_k^-(\max(s, x)) \\ &\leqslant F_{s[x]}(y) < G_{t[y]}(y) = G_t(y), \end{aligned}$$

by Lemma 12(i) and (ii). Note that

$$y \geqslant G_k^-\big(\max(M(s), x)\big) \geqslant G_k^-(\max(s, x)) > M(s[x]) \geqslant N(s[x]),$$

so $|s[x]| < |t[y]|$. $\square$

The following lemma will play the role of the "annihilator" in Robbin's work [8].

LEMMA 15. *If* $b_1, b_2, \ldots, b_j \leqslant n$ *and* $j \geqslant F_s(n + 2)$, *then*

$$\text{Zero } s[b_1, b_2, \ldots, b_j].$$

PROOF. By complete induction on $|s|$. If Zero s, this is trivial. Suppose not Zero s. If $j \geqslant F_s(n + 2) > F_{s[b_1]}(n + 2)$, then $j - 1 \geqslant F_{s[b_1]}(n + 2)$. So by complete induction (note $|s[b_1]| < |s|$ if not Zero s),

$$\left|s[b_1, b_2, \ldots, b_j]\right| = \left|s[b_1][b_2, \ldots, b_j]\right| = 0. \quad \square$$

As already indicated, the proof of the first stage (Theorem 18: if $|s| = |t|$ then $\mathfrak{S}_s \subseteq \mathcal{G}_t$ and $\mathfrak{S}_s \subseteq \mathfrak{F}_t$) will be founded on the preceding Lemmas 12–15. To complete the proof, we introduce notation to analyze the process of computing $E_s(b, a)$, for $s \in \mathbb{S}$ and $b, a \in \mathfrak{N}$. We will then define a predicate CT capable of recognizing correct computations of $E_s(b, a)$, suitably encoded. Lemmas 12–15 will serve to estimate the length of such computations, and so establish Theorem 18.

Let us write $(s, b, a) \to (t, d, c)$, where $s, t \in \mathbb{S}$ and $b, a, d, c \in \mathfrak{N}$, if Ind b, Ind a and one of the following six cases holds:

(i) Suc s, $b = \langle 0, 2\rangle$, $t = P(s)$, $d = (a)_0$ and $c = (a)_1$;

(ii) Lim s, $b = \langle 0, 2\rangle$, $t = s[(a)_{00}]$, $d = (a)_{01}$, and $c = (a)_1$;

(iii) $s = t$, $b = \langle 5, n, e, d_1, \ldots, d_m\rangle$, $m = (e)_1$, $d = d_i$ for some i ($1 \leqslant i \leqslant m$), and $a = c$;

(iv) $s = t$, $b = \langle 5, n, d, e_1, \ldots, e_m\rangle$, $m = (d)_1$, and $c = \langle el^H(e_1, a), \ldots, el^H(e_m, a)\rangle$ where $H = E_s$;

(v) $s = t$, $b = \langle 6, n, d\rangle$, $c = \langle i, (a)_1, (a)_2, \ldots\rangle = \mathrm{Re}(i, a)$ for some $i < (a)_0$;

(vi) $s = t$, $b = \langle 7, n, d\rangle$, and $c = \langle i, (a)_1, (a)_2, \ldots\rangle = \mathrm{Re}(i, a)$ for some $i < (a)_0$.

Intuitively, the relation $(s, b, a) \to (t, d, c)$ holds if one would naturally use $el^{E_t}(d, c)$ in the computation of $el^{E_s}(b, a)$. Let us say that (s, b, a) is *terminal* if there is no (t, d, c) such that $(s, b, a) \to (t, d, c)$. Informally, this means that $el^{E_s}(b, a)$ can be computed immediately, without recourse to any scratch computations $el^{E_t}(d, c)$. The triple (s, b, a) is terminal iff it falls under one of the following six cases:

(vii) not Ind b;

(viii) $b = \langle 0, 2\rangle$ and Zero s;

(ix) $b = \langle 1, n, q\rangle = \#(C_n^q)$, where $n \geqslant 1$;

(x) $b = \langle 2, n, i\rangle = \#(U_n^i)$, where $1 \leqslant i \leqslant n$;

(xi) $b = \langle 3, 2\rangle = \#(+)$;

(xii) $b = \langle 4, 2\rangle = \#(\dot{-})$.

Let $(s, b, a) \to^* (t, d, c)$ be the transitive closure of the relation $\to$; i.e., $(s, b, a) \to^* (t, d, c)$ iff there exists $n \geqslant 0$ and $(s_0, b_0, a_0), \ldots, (s_n, b_n, a_n)$ such that

$$(s, b, a) = (s_0, b_0, a_0) \to (s_1, b_1, a_1) \to \cdots \to (s_n, b_n, a_n) = (t, d, c).$$

In particular, $(s, b, a) \to^* (s, b, a)$ for all (s, b, a), even if (s, b, a) is terminal.

Lemma 16. *If* $(s, b, a) \to^* (t, d, c)$, *then*

(i) $el^K(d, c) < F_s^b(M(s) + a + k + 1)$, *where* $K = E_t$;
(ii) $d < F_s^b(M(s) + a + k + 1)$;
(iii) $c < F_s^b(M(s) + a + k + 1)$.

If $(s, b, a) \to^* (t, d, c)$ *and* Fin s, *then*
(iv) $el^K(d, c) < F_s^b(a + 1)$;
(v) $d < F_s^b(a + 1)$;
(vi) $c < F_s^b(a + 1)$.

PROOF. Throughout the proof we let $z = M(s) + a + k + 1, y = M(t) + c + k + 1$, $K = E_t$, $H = E_s$. The proof is by complete induction on $|s|$ and then on b. Now either (s, b, a) is terminal or it is not. We consider first the case (s, b, a) is terminal. In this case we are only considering $(s, b, a) \to^* (s, b, a)$; i.e., $(s, b, a) = (t, d, c)$. Trivially

$$d = b \leq F_s^b(a + 1) \leq F_s^b(M(s) + a + k + 1),$$
$$c = a \leq F_s^b(a + 1) \leq F_s^b(M(s) + a + k + 1).$$

We can establish the lemma by proving $el^H(b, a) \leq F_s^b(a + 1)$, for then $el^K(d, c) = el^H(b, a) \leq F_s^b(a + 1) \leq F_s^b(M(s) + a + k + 1)$. This is trivial in all six subcases, (vii)–(xii).

If there exists (t, d, c) such that $(s, b, a) \to (t, d, c)$, then by induction (since $(t, d, c) \to^* (t, d, c)$)

$$el^K(d, c) \leq F_t^d(M(t) + c + k + 1) = F_t^d(y),$$
$$d \leq F_t^d(y), \qquad c \leq F_t^d(y),$$

where y may be replaced by $c + 1$ if Fin t.

Suppose we can show that $F_t^d(y) \leq F_s^b(z)$ (if Fin s, that $F_t^d(c + 1) < F_s^b(a + 1)$). If $(s, b, a) \to (t, d, c) \to^* (u, f, e)$, then

$$el^J(f, e) < F_t^d(y) \leq F_s^b(z), \quad \text{where } J = E_u,$$
$$f < F_t^d(y) \leq F_s^b(z),$$
$$e \leq F_t^d(y) \leq F_s^b(z),$$

by induction. (If Fin s, y may be replaced by $c + 1$ and z by $a + 1$.) So to prove the induction step we will need

$$\left.\begin{aligned} el^K(d, c) < F_s^b(z) \\ d < F_s^b(z) \\ c < F_s^b(z) \end{aligned}\right\} \quad \text{for} \quad (s, b, a) \to (t, d, c),$$

$$\left.\begin{aligned} el^H(b, a) &\leq F_s^b(z) \\ b &\leq F_s^b(z) \\ a &\leq F_s^b(z) \end{aligned}\right\} \qquad \text{for} \quad (s, b, a) \to^* (s, b, a).$$

The last two of these are trivial. So we must show the other four and also $F_t^d(y) < F_s^b(z)$ (replace y by $c + 1$ and z by $a + 1$ if Fin s). The proof has six subcases, as listed in the definition of the relation $\to$. We will only prove the hardest subcases, which are (ii) and (iv); the others are proven similarly. The following proofs remain valid if y is replaced by $c + 1$ and z by $a + 1$ when Fin s, except as noted.

Subcase (ii): Lim s, $b = \langle 0, 2\rangle$, $t = s[(a)_{00}]$, $d = (a)_{01}$, $c = (a)_1$. Then $c < a$, and $d < a$.

Now if $|t| > |\bar{k}| = k$, then

$$\begin{aligned} y = M(t) + c + k + 1 &= M\big(s[(a)_{00}]\big) + c + k + 1 \\ &< F_{\bar{k}}(M(s) + a + k + 1) + a + k + 1 \\ &< F_{\bar{k}}^2(M(s) + a + k + 1) = F_{\bar{k}}^2(z). \end{aligned}$$

Let $w = F_{\bar{k}}^2(z)$. Then $w > y > M(t) + 1$ and $w > k$, so by Lemma 13

$$\begin{aligned} el^H(b, a) = el^K(d, c) &< F_t^d(y) < F_t^w(w) < F_s(w) \\ &= F_s F_{\bar{k}}^2(z) \leq F_s F_s^2(z) = F_s^3(z) < F_s^b(z). \end{aligned}$$

But if $|t| < k$, this argument does not apply, but rather we use the fact that Fin t:

$$\begin{aligned} el^H(b, a) = el^K(d, c) &< F_t^d(c + 1) < F_t^d(a + 1) < F_t^d(z) \\ &< F_t^w(w) < F_s(w) \leq F_s F_{\bar{k}}^2(z) < F_s F_s^2(z) < F_s^3(z) < F_s^b(z). \end{aligned}$$

Of course we also have (no matter what the value of $|t|$)

$$d \leq a < F_s^b(z), \qquad c < a < F_s^b(z).$$

Subcase (iv): $s = t$, $b = \langle 5, n, e, f_1, \ldots, f_m\rangle$, $m = (e)_1$, $d = e$ and $c = \langle el^H(f_1, a), \ldots, el^H(f_m, a)\rangle$. Then

$$\begin{aligned} c &< 3^m\big(el^H(f_1, a) + 1\big)^2 \cdots \big(el^H(f_m, a) + 1\big)^2 \\ &< 3^m\big(F_s^{f_1}(z) + 1\big)^2 \cdots \big(F_s^{f_m}(z) + 1\big)^2 \\ &\leq 3^m\big(F_s^{f_1+1}(z)\big)^2 \cdots \big(F_s^{f_m+1}(z)\big)^2 \\ &\leq 3^m F_s^{f_1+2}(z) \cdots F_s^{f_m+2}(z) \end{aligned}$$

by Lemma 10(ix). Let $w = \max(f_1 + 2, \ldots, f_m + 2) + 2m - 2 = \max(f_1, \ldots, f_m) + 2m$. Then by Lemma 10(viii)

$$c < 3^m F_s^{f_1+2}(z) \cdots F_s^{f_m+2}(z) < 3^m F_s^w(z).$$

But $3^m < 4^m = (F_{\bar{0}}(2))^m \leqslant F_{\bar{0}}^m(2) \leqslant F_s^m(2) \leqslant F_s^m(z)$, by Lemma 10(vi). So

$$c \leqslant 3^m F_s^w(z) \leqslant F_s^m(z) F_s^w(z) \leqslant F_s^{w+1}(z)$$

using Lemma 10(viii) again. Now

$$\begin{aligned} w + 1 &= \max(f_1, \ldots, f_m) + 2m + 1 \\ &\leqslant f_1 + \cdots + f_m + 2e + 1 \leqslant b, \end{aligned}$$

using Lemma 1(i). All together, $c \leqslant F_s^{w+1}(z) \leqslant F_s^b(z)$. Trivially, $d < b < F_s^b(z)$. And finally, we estimate

$$\begin{aligned} el^H(b, a) = el^K(d, c) &< F_t^d(y) = F_t^d(M(t) + c + k + 1) \\ &< F_s^d\big(M(s) + F_s^{w+1}(z) + k + 1\big) < F_s^d F_s F_s^{w+1}(z) \\ &= F_s^{d+w+2}(z). \end{aligned}$$

Now $d + w + 2 = e + 2m + \max(f_1, \ldots, f_m) + 2 < 3e + f_1 + \cdots + f_m + 2 < b$, using Lemma 1(i). So all together

$$el^H(b, a) = el^K(d, c) < F_t^d(y) \leqslant F_s^{d+w+2}(z) < F_s^b(z).$$

The proofs of subcases (i) and (iii) are trivial. Subcases (v) and (vi) require a trivial application of Lemma 10(viii) and (ix), respectively.

This completes the proof of the lemma. $\square$

For any given $s \in \mathbb{S}$ and $b, a \in \mathfrak{N}$, the calculation of $E_s(b, a)$ can be envisioned as proceeding along a branching tree. Thus if $b = \langle 5, n, e, f_1, \ldots, f_m \rangle$, where $m = (e)_1$, the tree branches into nodes for the calculations of $el^H(f_1, a), \ldots, el^H(f_m, a)$, and also for

$$el^H\big(e, \langle el^H(f_1, a), \ldots, el^H(f_m, a)\rangle\big),$$

where $H = E_s$. In general, there will be a node for each (t, d, c) such that $(s, b, a) \to^* (t, d, c)$.

Let us augment the nodes of the tree with a fourth component, so that the nodes are of the form $(t, d, c, el^K(d, c))$, where $K = E_t$. Usual coding techniques enable us to encode in $\mathfrak{N}$ such "computation trees". This can be done so that there is a primitive recursive predicate $CT(s, b, a, z)$ which recognizes the correct encoding z of the computation tree for $el^H(b, a)$, and so that there is a primitive recursive function $V(z)$ which extracts $el^H(b, a)$ from z.

To proceed further, we want an estimate of the size of z. If $(t, d, c, el^K(d, c))$ is a node of z, then $(s, b, a) \to^* (t, d, c)$, so $d, c, el^K(d, c) < F_s^b(M(s) + a + k + 1)$, by Lemma 16. Also $t = s[n_1, \ldots, n_k]$, where $n_1, \ldots, n_k < F_s^b(M(s) + a + k + 1)$, and

$$k \leqslant F_s\big(F_s^b(M(s) + a + k + 1) + 2\big)$$

by Lemma 15. Since Seq is primitive recursive, we can bound t primitive recursively in $F_s(F_s^b(M(s) + a + k + 1) + 2)$. The number of branches at the node $(t, d, c, el^K(d, c))$ is $\leqslant d \leqslant F_s^b(M(s) + a + k + 1)$. To bound the length of paths in the computation tree, we first notice that any path will consist of sections in which the first component remains constant, but the second component decreases. Since the second component is

$$\leqslant F_s^b(M(s) + a + k + 1),$$

this bounds the length of any section in which the first component remains constant. The number of maximal sections in a path is bounded by $F_s(F_s^b(M(s) + a + k + 1) + 2)$, by Lemma 15. The product of these two bounds must bound the length of any path. All together, we can define a primitive recursive function $A(u)$ such that

$$z \leqslant A(u) \quad \text{for all } u \geqslant F_s\big(F_s^b(M(s) + a + k + 1) + 2\big).$$

Thus we have the following "normal form" theorem:

LEMMA 17. *There exists a primitive recursive predicate $CT(s, b, a, z)$, and primitive recursive functions $V(z)$ and $A(u)$ such that for all $s \in \mathbb{S}$ and all $b, a \in \mathfrak{N}$, if $u \geqslant F_s(F_s(M(s) + a + k + 1) + 2)$, then*

$$el^H(b, a) = V(\mu z \leqslant A(u))CT(s, b, a, z),$$

where $H = E_s$.

Now we can prove:

THEOREM 18. *For all $s, t \in \mathbb{S}$, if $|s| = |t|$, then $\mathcal{E}_s \subseteq \mathcal{F}_t$ and $\mathcal{E}_s \subseteq \mathcal{G}_t$.*

PROOF. If Fin s, the result is known (Schwichtenberg [10]).

If Inf s, we prove only the harder case, if Lim s. Let $r = s[(b)_0]$, $H = E_r$, $d = (b)_1$, and $u = F_r(F_r^b(M(s) + a + k + 1) + 2)$. Then $E_s(b, a) = el^H(d, a) = V((\mu z \leqslant A(u))\, CT(r, d, a, z))$. Now

$$\begin{aligned} u = F_r\big(F_r^b(M(s) + a + k + 1) + 2\big) &\leqslant F_r^{b+2}(M(s) + a + k + 1) \\ &\leqslant F_r^{M(s)+a+b+k+1}(M(s) + a + b + k + 1) \\ &\leqslant F_s(M(s) + a + b + k + 1) \\ &\leqslant G_t G_{\bar{k}}(M(s) + a + b + k + 1) \\ &\leqslant F_t G_{\bar{k}}(M(s) + a + b + k + 1), \end{aligned}$$

by Lemmas 13 and 14. Hence the following is true, if we let
either $w = G_t G_{\bar{k}}(M(s) + a + b + k + 1)$
or $w = F_t G_{\bar{k}}(M(s) + a + b + k + 1)$:

$$E_s(b, a) = V\big((\mu z < A(u))CT(r, b, a, z)\big),$$

which is elementary in Seq, M, $G_{\bar{k}}$ and either G_t or F_t, depending on the choice of w. But Seq, M, $G_{\bar{k}} \in \mathfrak{P} \subseteq \mathcal{G}_t \cap \mathfrak{F}_t$, by Lemma 11(ii), since Inf t, so E_s is elementary in G_t and in F_t.

We now turn to the second stage of the proof that every p.r.-regulated system of notation has the subrecursive hierarchy equivalence property; i.e., proving the converse containments to Theorem 18. Our technique, using a form of the Recursion Theorem, is adapted from Schwichtenberg [10]. The only obstacle is that we are allowing Seq and N to be primitive recursive rather than only elementary. The next lemma provides certain functions for manipulating indices to get around this difficulty. They are all constructed by familiar techniques.

LEMMA 19. (i) *There is a primitive recursive function* $Gr(s, n)$ *such that if* $s \in \mathbb{S}$ *and* $|s| \geqslant n$, *then* $el^H(Gr(s, n), a) = F_{\bar{n}}((a)_0)$, *where* $H = E_s$. *I.e.*, $Gr(s, n) = \#_s(F_{\bar{n}})$.

(ii) *There is a primitive recursive function* $Gr^*(m, n)$ *such that if* $m > n$, *then* $el^H(Gr^*(m, n), a) = F_{\bar{n}}((a)_0)$, *where* $H = E_{\bar{m}}$. *I.e.*, $Gr^*(m, n) = \#_{\bar{m}}(F_{\bar{n}})$.

(iii) *There is a primitive recursive function* $Sh(s, n, b)$ *such that, for all* $s \in \mathbb{S}$ *and all* $n \in \mathfrak{N}$ *such that* $|s| \geqslant n$, $el^H(Sh(s, n, b), a) = el^K(b, a)$, *where* $H = E_s$ *and* $K = F_{\bar{n}}$. *I.e.*, $Sh(s, n, \#_{F_{\bar{n}}}(f)) = \#_s(f)$.

The next lemma is a Recursion Theorem, whose proof is readily generalized from that found in Schwichtenberg [10].

LEMMA 20. *If* $f(x, a_1, \ldots, a_m) \in \mathcal{E}_s$, *then there is an index* b *such that* $f(b, (a)_0, \ldots, (a)_{m-1}) = el^H(b, a)$, *where* $H = E_s$.

THEOREM 21. *If* $\mathbb{S}$ *is a p.r.-regulated system of notation, then for all* $s \in \mathbb{S}$, $\mathfrak{F}_s \subseteq \mathcal{E}_s$ *and* $\mathcal{G}_s \subseteq \mathcal{E}_s$.

PROOF. We will prove only the harder case, that $\mathfrak{F}_s \subseteq \mathcal{E}_s$. Choose $n > k$ so that Seq, Gr^*, Sh, Abs, $\mathrm{Char}_{\mathrm{Fin}} \in \mathfrak{F}_{\bar{n}}$. We will use the Recursion Theorem to construct a function $h(s) \in \mathfrak{F}_{\bar{n}}$ such that $F_s(x) = el^H(h(s), \langle x \rangle)$, for all $s \in \mathbb{S}$, where $H = E_s$.

We begin by listing the properties that will insure that $h(s) = \#_s(F_s)$. If Fin s, we can require

$$h(s) = Gr^*(\mathrm{Abs}(s), \mathrm{Abs}(s)).$$

If Suc s and Inf s, then (letting $K = E_{P(s)}$, $H = E_s = el^K$)

$$\begin{aligned} F_s(x) &= F_{P(s)}^x(x) = el^K\big(It\big(x, h(P(s))\big), \langle x \rangle\big) \\ &= H\big(It\big(x, h(P(s))\big), \langle x \rangle\big) \\ &= el^H\Big(Sb_1^2\big(\langle 0, 2 \rangle, \#_s\big(\lambda s.It\big(x, h(P(s))\big)\big), \#(\mathrm{In}_1)\big), \langle x \rangle\Big). \end{aligned}$$

(Recall $\mathrm{In}_1(x) = \langle x \rangle$; see Lemma 1(iii).) To determine $\#_s(\lambda x.It(x, h(P(s))))$, suppose h has index $\#_{F_{\bar{n}}}(h)$. Then $\#_s(h) = Sh(s, n, \#_{F_{\bar{n}}}(h))$, by Lemma 19(iii), so

$$\#_s\big(\lambda x.It\big(x, h(P(s))\big)\big) = Sb_1^2\Big(\#(It), \#\big(U_1^1\big), Sb_1^1\Big(Sh\big(s, n, \#_{F_{\bar{n}}}(h)\big), Sb_1^2\big(Sh\big(s, n, \#_{F_{\bar{n}}}(\mathrm{Seq})\big), \#\big(C_1^s\big), \#\big(C_1^0\big)\big)\Big)\Big).$$

Hence we should require

$$\begin{aligned} h(s) &= Sb_1^2\Big(\langle 0, 2\rangle, Sb_1^2\Big(\#(It), \#\big(U_1^1\big), Sb_1^1\Big(Sh\big(s, n, \#_{F_{\bar{n}}}(h)\big), \\ &\qquad Sb_1^2\big(Sh\big(s, n, \#_{F_{\bar{n}}}(\mathrm{Seq})\big), \langle 1, 1, s\rangle, \#\big(C_1^0\big)\big)\Big)\Big), \#(\mathrm{In}_1)\Big) \\ &= f_1\big(\#_{F_{\bar{n}}}(h), s\big) \quad \text{if Suc } s \text{ and Inf } s, \end{aligned}$$

where $f_1 \in \mathfrak{F}_{\bar{n}}$.

If Lim s, then $F_s(x) = F_{s[x]}\rho_s(x)$, where $\rho_s(0) = 0$, $\rho_s(1) = 1$, $\rho_s(x) = (\mu z)[z > \rho_s(x-1)$ and $(\forall y < x)[F_{s[y]}(z) < F_{s[x]}(z)]]$, if $x \geqslant 2$.

By Lemma 12(i), $\rho_s(x) \leqslant G_{\bar{k}}(\max(s, x))$. By Lemma 1(i),

$$\langle \rho_s(0), \rho_s(1), \dots, \rho_s(x)\rangle \leqslant 3^x\big(G_{\bar{k}}(\max(s, x)) + 1\big)^{2(x+1)} = D(s, x),$$

where $D \in \mathfrak{F}_{\bar{n}}$. This enables us to eliminate the recursion from the definition of ρ_s:

$$\begin{aligned} \rho_s(x) = \mathrm{Out}\Big(\big(\mu z \leqslant D(s, x)\big)\Big[(\forall i \leqslant x)\Big[&\text{if } i < 2 \text{ then } (z)_i = i, \text{ and if} \\ & i \geqslant 2 \text{ then } \big[(z)_i > (z)_{i-1} \text{ and} \\ & (\forall y < i)\big[F_{s[y]}((z)_i) < F_{s[i]}((z)_i)\big]\big]\Big]\Big], x\Big). \end{aligned}$$

In view of Lemma 2(iii) there is a function $f_2(b, s) \in \mathfrak{F}_{\bar{n}}$ such that

$$f_2\big(\#_{F_{\bar{n}}}(h), s\big) = \#_s(\rho_s),$$

assuming of course that $h \in F_{\bar{n}}$. So (letting $H = E_s$ and $K = E_{s[x]}$),

$$\begin{aligned} F_s(x) &= F_{s[x]}\rho_s(x) = el^K\big(h(s[x]), \langle \rho_s(x)\rangle\big) \\ &= E_s\big(\langle x, h(s[x])\rangle, \langle \rho_s(x)\rangle\big) \\ &= el^H\Big(Sb_1^2\Big(\langle 0, 2\rangle, Sb_1^2\Big(\#(\mathrm{In}_2), \#\big(U_1^1\big), Sb_1^1\Big(\#_s(h), \\ &\qquad Sb_1^2\big(\#_s(\mathrm{Seq}), \#\big(C_1^s\big), \#\big(U_1^1\big)\big)\Big)\Big), Sb_1^1\Big(\#(\mathrm{In}_1), \\ &\qquad Sb_1^1\big(f_2\big(\#_{F_{\bar{n}}}(h), s\big), \#\big(U_1^1\big)\big)\Big)\Big), \langle x\rangle\Big). \end{aligned}$$

So we should require

$$h(s) = Sb_1^2\Big(\langle 0, 2\rangle, Sb_1^2\Big(\#(\mathrm{In}_2), \#(U_1^1), Sb_1^1\big(Sh(s, n, \#_{F_{\bar n}}(h)), Sb_1^2\big(Sh(s, n, \#_{\bar n}(\mathrm{Seq})), \langle 1, 1, s\rangle, \#(U_1^1)\big)\big)\Big), Sb_1^1\big(\#(\mathrm{In}_1), Sb_1^1\big(f_2\big(\#_{F_{\bar n}}(h), s\big), \#(U_1^1)\big)\big)\Big)$$

$$= f_3\big(\#_{F_{\bar n}}(h), s\big) \quad \text{if Lim } s,$$

where $f_3 \in \mathfrak{F}_{\bar n}$. All together, we must require

$$h(s) = \begin{cases} Gr^*(\mathrm{Abs}(s), \mathrm{Abs}(s)) & \text{if Fin } s, \\ f_1\big(\#_{F_{\bar n}}(h), s\big) & \text{if Suc } s \text{ and Inf } s, \\ f_3\big(\#_{F_{\bar n}}(h), s\big) & \text{if Lim } s, \end{cases}$$

$$= f_4\big(\#_{F_{\bar n}}(h), s\big),$$

where $f_4 \in \mathfrak{F}_{\bar n}$. Let b be an index of $f_4(b, s)$ in $F_{\bar n}$, and let $h(s) = f_4(b, s)$. Then $h(s) = f_4(\#_{F_{\bar n}}(h), s)$, as required. Consequently, $el^H(h(s), \langle x\rangle) = F_s(x)$ for all $s \in \mathbb{S}$ and all $x \in \mathfrak{N}$. So $F_s \in \mathcal{E}_s$ for all $s \in \mathbb{S}$.

Similarly, $G_s \in \mathcal{E}_s$ for all $s \in \mathbb{S}$.

THEOREM 22. *If* $\mathbb{S}$ *is a p.r.-regulated system of notation, then* $\mathbb{S}$ *has the subrecursive hierarchy equivalence property.*

PROOF. By Theorems 18 and 21, $\mathcal{E}_s = \mathfrak{F}_s = \mathcal{G}_s$ for all $s \in \mathbb{S}$, and $\mathcal{E}_s = \mathcal{E}_t$ if $|s| = |t|$, for all $s, t \in \mathbb{S}$. The only thing remaining is to show that $\mathcal{E}_s = \mathfrak{F}_s^* = \mathcal{G}_s^*$ for all $s \in \mathbb{S}$. Now if $t <_{\mathbb{S}} s$, then $F_t, G_t \in \mathcal{E}_t \subseteq \mathcal{E}_s$, which establishes the result.

5. **Open problems.** This paper can only be a first contribution to the general problem of choosing suitable extensions of the Grzegorczyk Hierarchy. As such, it raises many open questions, all of which are variants of the informal question, "How good are regulation techniques?"

1. The most obvious approach is to consider systems of notation constructed analogously to Kleene's $\mathcal{O}$ [5], with the constraint of being regulated by some primitive recursive norm N. Thus, given an enumeration of the unary elementary functions, admit to $\mathcal{O}_N$ at each limit stage those elementary fundamental sequences which are regulated by N. Then $\mathcal{O}_N$ is p.r.-regulated. It appears to be difficult to find an N such that $\mathcal{O}_N$ demonstrably has a notation for ω^2. Assuming this is possible(2), one might hope to find an N which in fact contains notations for all "standard" fundamental sequences.

(2)The author now doubts that this is possible. It would seem that Feferman's technique in [2] can be adapted to show that $|\mathcal{O}_N| > \omega^2$ is contradictory.

Such a system might shed light on the elusive concept of "standard", both as it applies to ordinals and to subrecursive hierarchies.

2. What is the least ordinal which can not be represented in a p.r.-regulated system of notation? Is this the same as the least nonrecursive ordinal, or is it smaller? Is there a single p.r.-regulated system of notation with notations for every ordinal less than the least nonrepresentable ordinal?

3. What are the closure properties of the hierarchies of p.r.-regulated systems of notation?

4. Are there further conditions on p.r.-regulated systems of notation which will insure that, for any two such systems, their associated hierarchies are the same for all ordinals represented in both?

5. Are there weaker conditions than being p.r.-regulated, sufficient for the subrecursive hierarchy equivalence property?

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